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Nonlinear symmetry algebra of the MIC-Kepler problem on the sphere S^3

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Abstract. The quantum mechanical problem of motion in the dual charged Coulomb field modified by a centrifugal term (MIC-Kepler problem) is considered in the three-dimensional space of constant positive curvature S^3. Conserved operators in this problem form a cubic algebra similar to that of the Kepler problem on S^3. The explicit form of invariants of this algebra shows that its representation associated with the MIC-Kepler problem on S^3 is a nondegenerate one. The cubic symmetry algebra is used to obtain the energy spectrum of the problem.

1. Introduction

The study of the problem of motion in the dual charged Coulomb field with an additional inverse-square potential in flat space has a long history. It was independently introduced in [1, 2] and then intensively studied by Iwai and Uwano [3] both from the classical and quantum points of view. These authors were the first to use the term 'MIC-Kepler' for this problem. The geometric quantization approach to this problem has been discussed in [4]. The higher-dimensional generalization of the problem has also been considered [5]. One of the main points in these investigations has been to consider the symmetry group of the problem. It is known that the Kepler and MIC-Kepler problems in R^3 are quite similar. Both problems have an O(4, 2) group of the dynamical symmetry and the quantum mechanical Hamiltonians of these problems have a hydrogen-like form (see [6, 7]). Also, these problems have the same structure of the conserved operators which form the O(4) group.

In this paper we consider the MIC-Kepler problem in the three-dimensional spaces of constant curvature, and in particular on the sphere S^3. We show that the MIC-Kepler problem in these spaces possesses all these similarities with the Kepler problem on spaces of constant curvature. In particular, we show that the conserved quantum mechanical operators of the Runge–Lenz type, together with the generalized angular momentum operator, form a nonlinear (cubic) algebra similar to that of the Kepler problem on S^3. For this reason first of all we give a brief review of this last problem.

The quantum mechanical Kepler problem in a three-dimensional space S^3 of constant positive curvature was first considered by Schrödinger [8], and in the space H^3 of constant negative curvature by Infeld and Schild [9]. These authors found the energy spectrum to be degenerate, similar to that in flat space. An additional constant of motion, an analogue of the Runge–Lenz vector, which is the cause of this degeneration, was found in [10–12] for the problem on the sphere S^3, and in [13] for the Lobachevsky space H^3. As was noted in [12], these operators, together with angular momentum, generate an algebraic structure which may be considered as a nonlinear extension of the Lie algebra, and which was referred to in [14] as a cubic algebra.

Recently, the Kepler problem on the sphere S^3 has been used as a model for the description of quarkonium spectra [15], and exitons in quantum dots [16]. Many aspects of this problem in spaces S^3 and H^3, in particular separation of variables and path integral formulation, have been investigated in [14, 17–19].

We write the Schrödinger equation for the Kepler problem on the sphere S^3 as

$$H\psi = E\psi \qquad H = -\frac{1}{4R^2}M_{\mu\nu}M_{\mu\nu} - \frac{\alpha}{R}\frac{x_4}{|\boldsymbol{x}|} \tag{1}$$

where

$$\begin{aligned} &\boldsymbol{x} = \{x_1, x_2, x_3\} && M_{\mu\nu} = x_\mu\partial_\nu - x_\nu\partial_\mu \\ &x_\mu x_\mu = \boldsymbol{x}^2 + x_4^2 = R^2 && \mu, \nu = 1, 2, 3, 4 \end{aligned} \tag{2}$$

x_μ are coordinates in a four-dimensional flat space into which the sphere is embedded, and R denotes the radius of curvature. We use units such that $\hbar = m = 1$. Note that the operator $M_{\mu\nu}M_{\mu\nu}/2R^2$ coincides with the Laplacian operator on S^3 and $M_{\mu\nu}M_{\mu\nu}/2$ is the Casimir operator of the geometric O(4) group. Three generators, $-\mathrm{i}M_{ab}$ $(a, b = 1, 2, 3)$, constitute the angular momentum vector $\boldsymbol{L}$ and three generators, $-\mathrm{i}M_{a4} = P_a$, are the boost generators on the sphere. The spectrum of this problem is $E_n = -\alpha^2/2n^2 + (n^2 - 1)/2R^2$, $(n = 1, 2, 3, \ldots)$. The Hamiltonian H commutes with the angular momentum operator

$$L_a = -\mathrm{i}\epsilon_{abc}x_b\partial_c \qquad a, b, c = 1, 2, 3 \tag{3}$$

and with the analogue of the Runge–Lenz operators:

$$A_a = \frac{1}{2R}\epsilon_{abc}(L_bP_c - P_bL_c) + \frac{\alpha x_a}{|\boldsymbol{x}|}. \tag{4}$$

These operators form a nonlinear (cubic) algebra with an o(3) subalgebra generated by L_a:

$$\begin{aligned} &[A_a, A_b] = -2\mathrm{i}\left(H - \frac{L^2}{R^2}\right)\epsilon_{abc}L_c \\ &[L_a, A_b] = \mathrm{i}\epsilon_{abc}A_c \qquad [L_a, L_b] = \mathrm{i}\epsilon_{abc}L_c. \end{aligned} \tag{5}$$

Recently the algebras of this type have been intensively studied [20] in the context of symplectic reduction of Lie algebras and called finite W algebras by analogy with infinite-dimensional W algebras that appeared in conformal field theories. The algebra (5) is some deformation of the $so(4)$ algebra and has a coset structure $g_d = h + v_d$, where h is an o(3) algebra [21, 22]. The Casimir operators and some unitary irreducible representations for these algebras have been constructed in [22]. For the case of the algebra (5) the first and second Casimir operators in the notation of [22] are

$$\begin{aligned} &C_{1d} = a\boldsymbol{L}^2 + b\boldsymbol{L}^4 + \boldsymbol{A}^2 \qquad C_{2d} = \boldsymbol{L}\boldsymbol{A} \\ &a = -2H + \frac{2}{R^2} \qquad b = \frac{1}{R^2}. \end{aligned} \tag{6}$$

But from the expression (4) one can find that

$$\boldsymbol{A}^2 = 2H(\boldsymbol{L}^2 + 1) - \frac{1}{R^2}\boldsymbol{L}^2(\boldsymbol{L}^2 + 2) + \alpha^2 \qquad \boldsymbol{A}\boldsymbol{L} = \boldsymbol{L}\boldsymbol{A} = 0 \tag{7}$$

and therefore $C_{1d} = 2H + \alpha^2$ and $C_{2d} = 0$. Thus the Kepler problem in a space of constant curvature S^3 realizes some degenerate unitary irreducible representation of cubic algebra (5), similar to the way that the flat Kepler problem realizes a degenerate representation of O(4).

It will be shown in section 3 that the MIC-Kepler problem on the sphere S^3 realizes more general (nondegenerate) unitary representations of the cubic algebra (5).

This paper is organized as follows. In section 2 we consider the Dirac-like potential on S^3 as a solution of the Maxwell equations with radial magnetic field. Then we define a Hamiltonian of the MIC-Kepler problem on the sphere S^3 and obtain conserved Runge–Lenz type quantities which form a cubic algebra. By using this algebra in section 3 we obtain the spectrum of the MIC-Kepler problem on the sphere S^3.

2. The Hamiltonian of the MIC-Kepler problem on S^3

The most natural way to define the Dirac-like potential in a three-dimensional space of constant curvature is to solve the Maxwell equations in this space with a Coulomb-type magnetic field. The Dirac monopole in curved spaces has been considered in [24] where it was shown that the curvature of the background space plays no role in the quantization of the magnetic charge of the test particle.

Consider a dual charged test particle with unit mass, and charges (e_0, g_0) moving at nonrelativistic velocity on the sphere S^3 in the electric field $\vec{E}$ and magnetic field $\vec{H}$ of a stationary dual charged particle with charges (e, g) situated at the origin. We adopt the following abbreviations: $\alpha = (e_0 e + g_0 g)$ and $\mu = (e_0 g - e g_0)$. Quantization of the component of the angular momentum leads to the condition $\mu = n/2$ where n is an integer. At first we consider the case for which the charge $e = 0$ and therefore the electric field $\vec{E} = 0$.

It is convenient to use the four-dimensional spherical coordinates:

$$
\begin{aligned}
&x_1 = R\sin\chi\sin\theta\sin\phi \qquad x_2 = R\sin\chi\sin\theta\cos\phi \\
&x_3 = R\sin\chi\cos\theta \qquad\qquad\; x_4 = R\cos\chi \\
&0 \leqslant \chi \leqslant \pi \qquad 0 \leqslant \theta \leqslant \pi \qquad 0 \leqslant \phi \leqslant \pi .
\end{aligned}
\tag{8}
$$

The metric of sphere S^3 in these coordinates is

$$
\mathrm{d}s^2 = R^2(\mathrm{d}\chi^2 + \sin^2\chi\,\mathrm{d}\theta^2 + \sin^2\chi\sin^2\theta\,\mathrm{d}\phi^2). \tag{9}
$$

The Maxwell equations for the magnetic field $\vec{H}$ in these coordinates take the form

$$
\begin{aligned}
&\nabla\cdot\vec{H} = (R\sin^2\chi\sin\theta)^{-1}[\partial_\chi(\sin^2\chi\sin\theta H_\chi) + \partial_\theta(\sin\chi\sin\theta H_\theta) + \partial_\phi(\sin\chi H_\phi)] = 0 \\
&(\vec{\nabla}\times\vec{H})^\chi = (R\sin\chi\sin\theta)^{-1}[\partial_\theta(\sin\theta H_\phi) - \partial_\phi H_\theta] = 0 \\
&(\vec{\nabla}\times\vec{H})^\theta = -(R\sin\chi\sin\theta)^{-1}[\partial_\chi(\sin\theta\sin\chi H_\phi) - \partial_\phi H_\phi] = 0 \\
&(\vec{\nabla}\times\vec{H})^\phi = (R\sin\chi)^{-1}[\partial_\chi(\sin\chi H_\theta) - \partial_\theta H_\chi] = 0.
\end{aligned}
\tag{10}
$$

These equations have a Coulomb-type magnetic field solution:

$$
H_\phi = 0 \qquad H_\theta = 0 \qquad H_\chi = \mu/(R^2\sin^2\chi). \tag{11}
$$

Integration of the equations for the corresponding potential

$$
\vec{\nabla}\times\vec{A} = \vec{H} \qquad \vec{\nabla}\cdot\vec{A} = 0 \tag{12}
$$

gives the Dirac monopole-like potential as a particular solution:

$$
A_\chi = A_\theta = 0 \qquad A_\phi = \frac{\mu\tan\theta/2}{R\sin\chi}. \tag{13}
$$

The solution is valid everywhere except for the singularity line $\theta = \pi$ which connects points $\chi = 0$ and $\chi = \pi$. In fact, this solution describes the field of two magnetic charges with opposite signs situated at points $\chi = 0$ and $\chi = \pi$ and connected by the singularity line. It is worth noting that the electric Coulomb field considered by Schrödinger [8] is also created by two electric charges located at opposite points of S^3.

In the coordinates $\boldsymbol{x} = \{x_1, x_2, x_3\}$ the metric of the sphere is $\mathrm{d}s^2 = (\mathrm{d}\boldsymbol{x})^2 + (\boldsymbol{x}\,\mathrm{d}\boldsymbol{x})^2/(R^2 - \boldsymbol{x}^2)$ and the potential (13) has a form similar to that in R^3:

$$A_1(\vec{x}) = \mu\frac{-x_2}{|\boldsymbol{x}|(|\boldsymbol{x}| + x_3)} \qquad A_2(\vec{x}) = \mu\frac{x_1}{|\boldsymbol{x}|(|\boldsymbol{x}| + x_3)} \qquad A_3(\vec{x}) = 0. \tag{14}$$

Now we consider a corresponding quantum mechanical problem. The quantum-mechanical Hamiltonian of the motion of charged or dual charged particle in the monopole field is obtained by the substitution $\nabla_a \to \nabla_a + \mathrm{i}A_a$ in the Laplacian operator $\Delta = \nabla_a \nabla^a$:

$$H_A = -\tfrac{1}{2}(\nabla^a + \mathrm{i}A^a)(\nabla_a + \mathrm{i}A_a). \tag{15}$$

In order to have a more obvious analogy with the quantum mechanical Kepler problem in the spaces of constant curvature we will use four-dimensional notation. As we have seen (see the introduction), when $\alpha = 0$, the Hamiltonian (1) is proportional to the Casimir operator of the O(4) group which is $(L^2 + P^2)/2R^2$. Here $P_a = -\mathrm{i}(x_4\partial_a - x_a\partial_4)$ are boost operators on S^3. The natural generalization of this operator on sphere in the presence of the Dirac-type potential (14) is

$$N_a = x_4\pi_a - x_a p_4 \tag{16}$$

where the operators $\pi_a = -\mathrm{i}\partial_a + A_a$ and $p_4 = -\mathrm{i}\partial_4$ obey the following commutation relations:

$$\begin{aligned} &[\pi_a, x_b] = -\mathrm{i}\delta_{ab} \qquad &&[\pi_a, \pi_b] = \mathrm{i}\mu\epsilon_{abc}\frac{x_c}{|\boldsymbol{x}|^3} \\ &[\pi_a, p_4] = 0 &&[p_4, x_4] = -\mathrm{i}. \end{aligned} \tag{17}$$

By direct calculations it can be verified that the Hamiltonian (15) commutes with the generalized angular momentum vector:

$$J_a = \varepsilon_{abc}x_b\pi_c - \frac{\mu x_a}{|\boldsymbol{x}|}. \tag{18}$$

Now we rewrite the Hamiltonian H_A in a more convenient form. It should be noted that the presence of the Dirac-type potential (14) breaks the O(4) symmetry of the problem. The rhs of the commutator of two operators N_a contains a term proportional to the field strength and therefore operators J_a, N_a do not form an o(4) algebra:

$$[N_a, N_b] = \mathrm{i}\varepsilon_{abc}J_c + R^2F_{ab} \qquad [J_a, N_b] = \mathrm{i}\varepsilon_{abc}N_c \qquad [J_a, J_b] = \mathrm{i}\epsilon_{abc}J_c \tag{19}$$

where $F_{ab} = [\pi_a, \pi_b]$ is given in (17). But in spite of this the Hamiltonian H_A can be presented in the form similar to that of the Hamiltonian of the Kepler problem (1) for $\alpha = 0$:

$$H_A = \frac{J^2 + N^2}{2R^2} - \frac{\mu^2}{2R^2}. \tag{20}$$

The spectrum of this Hamiltonian obtained from the solution of the Schrödinger equation depends on the eigenvalues of J^2. Therefore it is clear that, besides J_a (18), there are no additional quantities that commute with the Hamiltonian (20). For this reason we consider a modification of this problem.

In the analogy with the flat case, we introduce a Hamiltonian with a Zwanziger-like term $\mu^2/2|\boldsymbol{x}|^2$:

$$H_\mu = H_A + \frac{\mu^2}{2|\boldsymbol{x}|^2} = \frac{J^2 + N^2}{2R^2} + \frac{\mu^2 x_4^2}{2R^2|\boldsymbol{x}|^2}. \tag{21}$$

Then, using commutational relations (17) and (19) we find that

$$[H_\mu, N_a] = \mathrm{i}\mu\frac{x_4 J_a}{|\boldsymbol{x}|^3}. \tag{22}$$

By using this relation one can check that the Hamiltonian (21) commutes with the generalized angular momentum operator (18) and additional constant of motion

$$\tilde{A}_a = \frac{1}{2R}\epsilon_{abc}(J_b N_c - N_b J_c). \tag{23}$$

Operators $\tilde{A}_a$ and J_a obey the following commutation relations:

$$\begin{aligned} &[\tilde{A}_a, \tilde{A}_b] = -2\mathrm{i}\left(H_\mu - \frac{J^2}{R^2} + \frac{\mu^2}{2R^2}\right)\epsilon_{abc}J_c \\ &[J_a, \tilde{A}_b] = \mathrm{i}\epsilon_{abc}\tilde{A}_c \qquad [J_a, J_b] = \mathrm{i}\epsilon_{abc}J_c. \end{aligned} \tag{24}$$

Now we add the Coulomb term to the Hamiltonian (21) and finally obtain the Hamiltonian of the MIC-Kepler problem on the sphere:

$$H_\alpha = \frac{J^2 + N^2}{2R^2} + \frac{\mu^2 x_4^2}{2R^2|\boldsymbol{x}|^2} - \frac{\alpha x_4}{R|\boldsymbol{x}|}. \tag{25}$$

Then we verify that this Hamiltonian commutes with the operators of generalized angular momenta (18) and with the analogue of the Runge–Lenz vector

$$A_a = \frac{1}{2R}\epsilon_{abc}(J_b N_c - N_b J_c) + \frac{\alpha x_a}{|\boldsymbol{x}|}. \tag{26}$$

These operators satisfy the commutational relations of the cubic algebra:

$$\begin{aligned} &[A_a, A_b] = -2\mathrm{i}\left(H_\alpha - \frac{J^2}{R^2} + \frac{\mu^2}{2R^2}\right)\epsilon_{abc}J_c \\ &[J_a, A_b] = \mathrm{i}\epsilon_{abc}A_c \qquad [J_a, J_b] = \mathrm{i}\epsilon_{abc}J_c. \end{aligned} \tag{27}$$

Furthermore, the following equalities hold:

$$\boldsymbol{A}^2 = 2H_\alpha(\boldsymbol{J}^2 - \mu^2 + 1) - \frac{1}{R^2}\boldsymbol{J}^2(\boldsymbol{J}^2 - \mu^2 + 2) + \alpha^2 \qquad \boldsymbol{AJ} = \boldsymbol{JA} = -\alpha\mu. \tag{28}$$

The deformed Casimir operators (see the introduction) for the algebra (27) are

$$\begin{aligned} &C_{1d} = c\boldsymbol{J}^2 + d\boldsymbol{J}^4 + \boldsymbol{A}^2 \qquad C_{2d} = \boldsymbol{JA} = \boldsymbol{AJ} \\ &c = -2H_\alpha + \frac{(2-\mu^2)}{R^2} \qquad d = \frac{1}{R^2} \end{aligned} \tag{29}$$

and therefore $C_{1d} = 2H_\alpha(1-\mu^2) + \alpha^2$, $C_{2d} = -\alpha\mu$. Thus the representation of the cubic algebra realized by the MIC-Kepler problem on the sphere S^3 is nondegenerate.

Note that for $\mu = 0$ the MIC-Kepler problem becomes the Kepler problem on the sphere S^3. Let us take the limit $R \to \infty$ keeping x_a finite and $x_4/R \to 1$. The Hamiltonian (25) then becomes the Hamiltonian of the flat MIC-Kepler problem. Since in this limit we have $N_a/R \to -\mathrm{i}p_a + A_a$, the algebra (27) reduces to the o(4) algebra generated by the generalized angular momentum operator and Runge–Lenz operator of the flat MIC-Kepler problem. Also equalities (28) go over into those for the flat problem and the Casimir invariants C_{1d}, C_{2d} transform into the invariants C_1, C_2 of the o(4) algebra of the flat MIC-Kepler problem.

3. The spectrum of the MIC-Kepler problem on the sphere

In this section we show that, by using the relations (27) and (28), one can find the spectrum of the MIC-Kepler problem on the sphere. This algebraic treatment of the MIC-Kepler problem in spaces of constant curvature is based on the approach commonly used for obtaining infinitesimal operators of the unitary representations of the proper Lorentz group (see [25]). A similar approach was applied for nonlinear algebras in papers [11, 12, 19, 21, 22].

Let us introduce the linear combinations of the operators A_a and J_a: $J_\pm = J_1 \pm \mathrm{i}J_2$ and $A_\pm = A_1 \pm \mathrm{i}A_2$. Then the commutation relations of the algebra (27) take the form

$$\begin{aligned}
&[A_\pm, A_3] = \pm 2\left(H_\alpha - \frac{J^2}{R^2} + \frac{\mu^2}{2R^2}\right) J_\pm \\
&[A_+, A_-] = -4\left(H_\alpha - \frac{J^2}{R^2} + \frac{\mu^2}{2R^2}\right) J_3 \\
&[J_\pm, A_3] = \mp A_\pm \qquad [J_\pm, A_\mp] = \pm 2A_3 \qquad [J_3, A_\pm] = \pm A_\pm \\
&[J_\pm, A_\pm] = [J_3, A_3] = 0 \qquad [J_\pm, J_3] = \mp J_\pm \qquad [J_+, J_-] = 2J_3.
\end{aligned} \tag{30}$$

Let ψ_{Ejm} denote the common eigenfunctions of operators H, J^2, J_3 with eigenvalues E, $j(j+1)$, m, respectively. By analogy with [25], we then find from (30) that the operators $A_\pm$, A_3 are given by

$$\begin{aligned}
A_\pm \psi_{Ejm} = {} & \pm C_j\sqrt{(j \mp m)(j \mp m - 1)}\psi_{E,j-1,m\pm1} \\
& \pm C_{j+1}\sqrt{(j \pm m + 1)(j \pm m + 2)}\psi_{E,j+1,m\pm1} \\
& - B_j\sqrt{(j \mp m)(j \pm m + 1)}\psi_{E,j,m\pm1} \\
A_3\psi_{Ejm} = {} & C_j\sqrt{j^2 - m^2}\psi_{E,j-1,m} - C_{j+1}\sqrt{(j+1)^2 - m^2}\psi_{E,j+1,m} - mB_j\psi_{Ejm}
\end{aligned} \tag{31}$$

where B_j, C_j do not depend on m. For operators $J_\pm$, J_3 we have the conventional relations

$$J_\pm\psi_{Ejm} = \sqrt{(j \mp m)(j \pm m + 1)}\psi_{E,j,m\pm1} \qquad J_3\psi_{Ejm} = m\psi_{Ejm}. \tag{32}$$

From equations (30)–(32) the recurrence relations for B_j and C_j follow:

$$[(j+2)B_{j+1} - jB_j]C_{j+1} = 0 \tag{33}$$

$$(2j-1)C_j^2 - (2j+3)C_{j+1}^2 - B_j^2 = 2\left(E - \frac{j(j+1)}{R^2} + \frac{\mu^2}{2R^2}\right). \tag{34}$$

Pursuing further the analogy with representations of the Lorentz group [25], let $j_0 \geqslant 0$ denote the lowest value of angular momentum belonging to the representation space of the algebra (27). From equations (31) we can see that this definition is equivalent to

$$C_{j_0} = 0 \qquad C_{j_0+1} \neq 0. \tag{35}$$

Introducing the notation $j(j+1)B_j = \tau_j$, we find from equation (33) $\tau_{j+1} - \tau_j = 0$, that is, τ_j does not depend on j. Denoting this constant by j_0c, we find

$$B_j = \frac{j_0c}{j(j+1)}. \tag{36}$$

Now, introducing the notation $(2j-1)(2j+1)C_j^2 = \sigma_j$, we obtain from equation (34)

$$\sigma_j - \sigma_{j+1} = 2(2j+1)\left(E - \frac{j(j+1)}{R^2} + \frac{\mu^2}{2R^2}\right) + j_0^2c^2\left(\frac{1}{j^2} - \frac{1}{(j+1)^2}\right) \tag{37}$$

and, as a consequence,

$$\sigma_{j_0} - \sigma_j = \sum_{k=j_0}^{j-1}(\sigma_j - \sigma_{j+1}) = (j^2 - j_0^2)\left(2E - \frac{j^2 + j_0^2 - 1}{R^2} + \frac{\mu^2}{R^2} + \frac{c^2}{j^2}\right). \tag{38}$$

Since $\sigma_{j_0} = 0$, we arrive at

$$C_j^2 = -\frac{j^2 - j_0^2}{4j^2 - 1}\left(2E + \frac{c^2}{j^2} - \frac{j^2 + j_0^2 - 1}{R^2} + \frac{\mu^2}{R^2}\right). \tag{39}$$

Using equations (31), (32) and taking into account equation (36) we find $(\boldsymbol{AJ})\psi_{Ejm} = (-j_0c)\psi_{Ejm}$. Therefore equation (28) yields

$$cj_0 = \alpha\mu. \tag{40}$$

From equation (31) we find

$$\boldsymbol{A}^2\psi_{Ejm} = [-j(2j-1)C_j^2 - (j+1)(2j+3)C_{j+1}^2 + j(j+1)B_j^2]\psi_{Ejm}. \tag{41}$$

Taking into account equations (36), (39) and (40) we arrive at

$$\boldsymbol{A}^2\psi_{Ejm} = \left(2E(j^2+j-j_0^2+1) - \frac{j(j+1)(j^2+j-j_0^2+2)}{R^2} + \frac{(\mu^2-j_0^2)(j^2+j-j_0^2+1)}{R^2} + c^2\right)\psi_{Ejm}. \tag{42}$$

Then by comparing equations (28) and (42) we find that $c^2 = \alpha^2$ and $j_0^2 = \mu^2$. Thus the final expression for C_j^2 (see (39)) is

$$C_j^2 = -\frac{(j^2-\mu^2)[2Ej^2R^2 - j^2(j^2-1) + \alpha^2R^2]}{R^2j^2(4j^2-1)}. \tag{43}$$

Due to the the quantization condition for magnetic charge $\mu = \pm 0, \pm\frac{1}{2}, \pm 1, \ldots$ one can identify: $j_0 = |\mu|$ and $c = \alpha\mu/|\mu|$.

The next step consists in taking into account the condition requiring that conserved operators be Hermitian. From this requirement, it follows that the coefficients C_j must satisfy the conditions

$$C_0 = C_0^\star \tag{44}$$

$$C_j = -C_j^\star \qquad j \geqslant 1. \tag{45}$$

It can be seen from (43) that condition (44) is satisfied identically. In the space S^3, condition (45) is satisfied only if (i) $j \geqslant |j_0| = |\mu|$ and (ii) if for a fixed value of E, quantum number j is bounded from above, that is, $j \leqslant j_{\max}$. Denoting $j_{\max}+1 = N$, we obtain $C_N = 0$, that is (see (43)), $[2E_N - (N^2-1)/R^2 + \alpha^2/N^2] = 0$ and $N > |\mu|$, whence it follows that the energy levels are given by

$$E_N = -\frac{\alpha^2}{2N^2} + \frac{N^2-1}{2R^2} \qquad N = |\mu|+1, |\mu|+2, |\mu|+3, \ldots. \tag{46}$$

When $\mu = 0$ this spectrum coincides with the spectrum of the Kepler problem on the sphere S^3, and when $R \to \infty$ the spectrum (46) goes over into the spectrum of the flat MIC-Kepler problem.

4. Conclusion

The above considerations show that the quantum mechanical MIC-Kepler problem on the sphere S^3 has additional Runge–Lenz-type conserved quantities. These operators, together with generalized angular momentum, form a nonlinear (cubic) algebra.

It was also shown that the two (deformed) Casimir invariants of this algebra are nonzero. Therefore the MIC-Kepler problem on S^3 realizes nondegenerate unitary irreducible representations of the cubic algebra.

By using cubic algebra defined by equations (27)–(29) one is able to obtain the spectrum of the MIC-Kepler problem in the space S^3.

We note that the spectrum of the MIC-Kepler problem in the space H^3 can be obtained by the same method. The expression for the spectrum in this space is obtained by the formal substitution $R \to \mathrm{i}\rho$, where ρ is a real number.

The spectra obtained by algebraic consideration coincide with those obtained by solution of the Schrödinger equation in these spaces.

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